EDUCATION IN THE COMMONWEALTH

MATHEMATICS TEACHING IN SCHOOLS

COMMONWEALTH SECRETARIAT

This book is addressed to serving teachers, student teachers and their tutors, curriculum developers, supervisors, inspectors and others, who are concerned with the teaching of mathematics in schools and are looking for ideas and suggestions for making mathematical education interesting, attractive, live and fruitful. It is written at the instance of the Commonwealth Secretariat and is based on Mathematics in Commonwealth Schools, the Report of a Specialist Conference held in the University of the West Indies, Trinidad, in September, 1968.

The main topics in the book deal with mathematics in primary schools and in secondary schools, assessment and evaluation, the training of teachers, and books and other resources for learning mathematics. Each topic is dealt with in a self-contained chapter consisting of a discussion of the topic, some suggestions for action, and a short list of books that could usefully be consulted by anyone having a special interest in the topic.

Grateful acknowledgement is made to the Chairman of the Conference, the Lead Speakers, the Chairmen of the Working Groups and to all other delegates and observers, whose papers, reports, and ideas I have quoted, adopted, adapted, modified or used in any way. I hope many readers will find the time to read the main Report and the books by some of the speakers and delegates. Acknowledgement to other authors is made in the footnotes.

Next, I acknowledge the co-operation of many friends in connection with my visits to schools and enquiries of recent developments in their respective countries. More particularly, I acknowledge the help of Mrs E. M. Williams, C.B.E., and Miss E. E. Biggs, H.M.I., the former for taking me round a number of English schools, a teachers' centre and a College of Education and the latter for showing me a sample collection of children's work in mathematics. Both authors are well-known for their expertise in the teaching of primary school mathematics. The influence of their writings and presentations will be apparent in Chapter 6 but it may also be seen in practically all the other chapters.

Finally, I thank the Commonwealth Secretariat for the opportunity and Mr. Stephen O. Sanusi for typing the script.

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Number Ten

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1. INTRODUCTION

No teacher of mathematics can afford to ignore the changes that are taking place in mathematical education in the schools. These changes concern method and content and are calculated to improve the mathematical education of all children, having regard to their varying abilities. Mathematics is a basic tool in the development of science, technology, commerce and industry and hence in the economic development of a modern society. Mathematics is also a way of thinking and reasoning which enhances the education of a man, no matter what his place in society. All children therefore need to be equipped with this essential tool of productiveness and thought, to varying extents according to their abilities, so that in time they may make their contribution to the economy and the government of their societies and also improve their own qualities as human beings. In furthering this cause, the role of a teacher is to assist the children in learning mathematics. It is the purpose of the following pages to show how to perform that role efficiently.

The first step is a clear statement of the aims of mathematical education which is relevant to the needs of the society and the individuals. The aims resolve into the two strands of usefulness and personal development. It should be the aim of mathematical education to enhance the usefulness of an individual in his contribution to the economic development of society and the economic stability of himself. It is also a desirable aim of mathematical education to assist the individual to think straight, solve his problems and function effectively as a citizen of a civilised state. At the beginning of a new topic or lesson, the teacher should ask himself how far the topic or lesson promotes one or another of the aims of mathematical education. These aims and the fundamental ideas of mathematical education are elaborated in Chapter 2.

Primary education is the foundation on which any superstructure is laid. Considerable space is therefore devoted in Chapter 3 to the method and content of mathematical education in the primary school. The change in method is less of direct teaching and more of providing experience and opportunities for children to discover facts and relations in mathematics. The children should be allowed to experiment, investigate and draw conclusions based on their own findings and expressed in their own words.

In this approach, the environment and the culture are vitally important. The children should be assisted to see mathematics in their environment and as an inherent factor of their culture. It is from the familiar surroundings and base that mathematical abstractions are made and purified. A teacher should select the experience and topics which assist children best in learning mathematics in that exploratory manner.

In the secondary school, the mathematics teacher is a specialist in mathematics. He must know his subject and be able to interpret the syllabus with a view to promoting mathematics learning at that level. He should keep in view the needs of the future mathematicians who would create more mathematics, the future engineers, biologists, chemists, physicists, agriculturists and many other scientists and technologists who need mathematics for their work, the intermediate level workers and technicians and a large number whose mathematical education ends in the secondary school. Chapter 4 contains statements and suggestions which should help a teacher to pursue these objectives of mathematical education more efficiently.

The last three chapters deal respectively with the assessment of children's progress and evaluation of programmes, the training and re-training of teachers and the resources for learning mathematics. The importance of a competent teacher cannot be over-stressed. A competent teacher, trained and re-trained, armed with suitable resources for learning mathematics and equipped with the latest methods of assessment and evaluation, is a valuable asset in the promotion of mathematical education in schools. It is suggested that it should be the aim of every teacher to become a competent teacher and that this book should help in achieving that aim.

2. FUNDAMENTAL IDEAS AND OBJECTIVES OF MATHEMATICAL EDUCATION[1]

Objectives

The objectives of the school fall into two strands. The first is the narrow but popular aim of enabling the child to have a job or earn a living when he leaves school. On this there is general agreement by the state, the family, the neighbours, the public and sometimes even the child himself. But this aim should not be interpreted too narrowly. We do not know sufficiently well in advance what job is the most suitable for the child to undertake or what job he will in fact get. The school should therefore aim at developing the child widely so as to present him with many opportunities of employment from which he can make his choice. The need for a general preparation leads to the second and wider objectives of the school, which are to develop all the inner resources of the child; that is, to give him culture.

Mathematical education similarly has its narrow and wider objectives. The narrow objectives are that the school should equip the child with sufficient mathematical education which will make him employable. Even the more lowly forms of employment make some demand on mathematical education. The more modest aim of primary school mathematical education is

(1) arithmetic of natural numbers (+, -, x, ÷),

(2) length, area, volume,

(3) fractions, percentage

There is no doubt that the primary school should teach arithmetic. For all practical purposes, it is sufficient that children do their arithmetic mechanically, having regard for accuracy and some speed. It is desirable, however, that the children should also learn to do their arithmetic insightfully so that they get the results faster and obtain more permanent results.

The narrow objectives of mathematical education at the secondary school level are that the school should assist in the professional preparation of prospective technical personnel such as technicians, technologists, engineers, surveyors, scientists, managers. Mathematics above the primary school level is strictly mandatory for certain kinds of technical personnel and immediately necessary only for prospective users of mathematics. For the same reason that it cannot be decided far ahead what kind of employment will be most suitable for the secondary school leaver or what employment he will actually get, every child entering the secondary school ought to receive mathematical education which is aimed at general competence for a variety of employments.

These narrow objectives are desirable but they are not enough. Of course, they could actually lead to the wider objectives and higher ideals, namely, to

(1) serve the individual and the community

(2) develop all the inner resources of the child.

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[1] Adapted from the lead paper by Professor G. Polya.

Several of such general objectives of education have been proposed, from Plato downward, in various formulations. Here are a few:

(1) General culture

(2) Discipline of the mind

(3) Desirable habits of thinking

(4) Mental and emotional maturity

(5) A well balanced personality.

Each aim mentioned embodies some high ideal worth striving for. Yet every one of the aims may be worthless if only lip service is paid to it and it may be worse than worthless if it is used as a cheap and empty slogan. On the other hand, each of those ideals is worth while provided that one honestly believes in it and it is really valuable if one earnestly tries to live up to it and translate it into everyday practice of the school. In planning the curriculum or preparing a lesson or choosing a problem for homework, therefore, it is desirable to ask oneself "What is it good for?" If it does not promote any objective, narrow or wider, it is not worth undertaking.

Fundamental Ideas

Here are two simple rules on the process of learning. The first is that learning should be active. As expressed in a picturesque metaphor by Socrates: the ideas should be born in the student's mind and the teacher should act only as midwife. The teacher must "let the students discover by themselves as much as feasible under the given circumstances." The second rule is that there are consecutive phases in the process of learning. Things should come before words, the concrete before the abstract, doing and seeing before verbal expression and so on. In down-to-earth language, learning begins with action and perception, proceeds from thence to words and concepts and should end in desirable mental habits.

There is general agreement that mere possession of information is of little value in mathematics. To know mathematics means to be able to do mathematics, use mathematical language, find the unknown, check a proof and so on. Therefore, to teach mathematics, the student must be given the opportunity to do mathematics. There are many ways of doing mathematics but solving problems appears as the most cardinal mathematical activity. Hence it is well justified that all mathematical textbooks contain problems. The problems may even be regarded as the most essential contents of a textbook and problem-solving by the students as the most essential part of mathematical instruction.

Problems may be routine or non-routine. A routine problem has a specific aim. It should teach the student to use correctly this or that particular rule, procedure or definition. It offers drill and practice but does not demand any invention or originality. To solve a quadratic equation with given numerical coefficients is a routine problem for a student who has been shown the general formula. On the contrary, a non-routine problem challenges the student's inventiveness and originality and should aim at some more general and higher objective.

Mathematical problems should be used to implant in the minds of the students whatever attitudes and procedures may be generally useful for solving any kind of problem. This is what Polya calls the tactics of problem-solving.

In teaching problem-solving in the mathematics class, one quite naturally comes across attitudes and patterns of thought, the usefulness of which is not restricted to mathematical problems. For example, in solving a problem the right attitude is first of all to understand the problem, distinguish its principal parts and see each of them as clearly as one can and then try to foresee the result. Above all, one should try to conceive a plan before diving into details. What is described here is an attitude whose usefulness extends far beyond elementary mathematical problems. This attitude belongs to the tactics of problem-solving. Here is one procedure, mentioned in many of the older textbooks of geometry. We begin by taking for granted the result, the conclusion that we have to prove or the figure that we have to construct. Then we derive from it some other conclusion or figure, hence still another one, and so on, until we arrive at the hypothesis or the data proposed. This procedure of "working backwards", familar in problems of construction in geometry is not restricted to geometry, but has a wider interest[1]; it belongs in fact to the tactics of problem-solving.

Some of the students will, others will not, use mathematics after leaving the secondary school. For users of mathematics the tactics of problem-solving may be the beginning of their professional attitude. For non-users it may easily be the most useful thing that remains with them from the mathematics of the secondary school.

In conclusion, before giving a lesson, the teacher should ask himself "What is it good for? Does it promote any narrow or wider objective?" Unless it promotes some objective, especially the wider objectives and the higher ideals, it is not worth giving.

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1 For further discussion, see G. Polya: How to Solve It, New York, Doubleday & Co., 1957, pp. 225-232.

Selected Reading List

The views of G. Polya are further developed and illustrated in his books. Other authors express the same or similar ideas in different words. Some pursue the two broad strands of objectives along ramifying channels. A teacher should read these objectives and the fundamental ideas, not only at the beginning of a programme but from time to time, so as to maintain a clear view of the objectives and grasp the ideas and the approach which lead to successful mathematical education.

The list is by no means exhaustive as is evident from the bibliographies in some of the books.

1. Biggs, E.E. and Maclean, J.R.: Freedom to Learn, Ontario, Addison-Wesley (Canada) Ltd., 1969.

2. Commonwealth Secretariat: Mathematics in Commonwealth Schools, London, Commonwealth Secretariat, 1969.

3. Dienes, Z.P.: Building Up Mathematics, London, Hutchinson Educational Ltd., 1964.

4. Hardy, G.H.: A Mathematician's Apology, Cambridge, Cambridge University Press, 1940.

5. Land. F.W. (ed.): New Approaches to Mathematics Teaching, London, Macmillan, 1966.

6. Peel, E.A.: The Pupil's Thinking, London, Oldbourne, 1967.

7. Polya, G.: How to Solve It, New York, Doubleday, 1957.

8. Polya, G.: Mathematics and Plausible Reasoning (2 vols.), Princeton, Princeton University Press, 1954.

9. Polya, G.: Mathematical Discovery (2 vols.), London, John Wiley and Sons, 1962/65.

10. Skemp, R.R.: Understanding Mathematics, London, University of London Press, 1964.

11. Wertheimer, M.: Productive Thinking, New York, Harper, 1945.

12. Whitehead, A.N.: The Aims of Education, London, Ernest Benn Ltd., 1962.

3. MATHEMATICS IN PRIMARY SCHOOLS[1]

The teaching of mathematics in the primary school by the new approach, otherwise known as "discovery" learning, is a challenging and rewarding experience for children and their teacher. Primary school children, whose ages in most countries range from five to twelve years, learn to discover for themselves mathematical concepts, patterns and relationships and to solve problems in mathematics. In countries like Britain, where syllabuses are not externally imposed and head teachers are free to devise their own schemes of work and classroom methods, it is easy for teachers to try the new approach, which consists of getting children to work in small groups or individually, encouraging them to think for themselves and investigate mathematical problems, which they do in individual and sometimes highly original ways.

The aims of discovery learning in mathematics are

(i) to set children free to think for themselves

(ii) to give them opportunities to discover the order and pattern, which is the essence of mathematics

(iii) to give them the skills.

To promote discovery learning, the teacher should know the environment and be able to select and structure his programme. He should plan the learning of each child so that the child could discover all the mathematics and mathematical skills he should learn. The teacher should provide the right experience and ask the questions which would encourage the child to discover his own methods in written calculation as well as in making discoveries with shapes and patterns.

Discovery learning is a method that can and should be used in every aspect of mathematics, at every stage and with children of all abilities. Differences in abilities should be recognised. Some children grasp abstract ideas easily and after only a brief acquaintance with real materials; others require a lengthy period with real materials before they are able to abstract a single idea or concept. Some respond to open-ended questions, others need directed questions. It is the teacher's responsibility to provide the right experience for the child and ask him suitable questions that would enable him to take the final step and make the discovery himself. When a teacher finds himself telling the answer, he should admit that either the child is not ready for the experience or the teacher is not asking the right questions,

Here are two examples of questions, one directed and the other open-ended, in Miss Biggs' own words:

> For several years I had given considerable direction to children and teachers in isolating the variables when working with a pendulum. For example, one assignment was: "Time the pendulum for 30 swings for lengths 6", 12", 18" as far as 48". Draw a graph. Can you find from your graph the length of a pendulum which beats seconds?" A group of ten year olds performed the experiment carefully and obtained a reasonable result. A month later I discussed the pendulum with the same group. "What did you discover?",

[1] Adapted from the lead paper by Miss E.E. Biggs, H.M.I.

I asked. Their recollections were so hazy and confused that we we had to start at the beginning once more.

Contrast this experience with the next, in which the question was open-ended:

> Last summer I was working with a group of nine and ten year old boys in down-town New York. We had decided to time various objects which the boys had chosen to roll down a long slope in the corridor. We had no stop watch, so I had added a length of fine string and a piece of plasticine to our collection. When the boys asked for a stop watch, I asked them if they could devise a means of timing from the materials I had provided. On seeing these Richard immediately suggested making a pendulum. It so happened that there was a large hook fixed at a height of 7 feet above the slope, to which the boys attached the longest pendulum they could make. "Where shall we start?" I asked. "Up at the ceiling, straight out", Earl replied. "Does it matter where we start?" I questioned. They decided that it did matter and we set the pendulum swinging. "Does it beat regularly?" I asked. "Let's count," they replied. But the boys decided that the pendulum was swinging too slowly for effective timing. "How shall we change the beat?", I asked. "Shorten the string," suggested Richard. "Lengthen the string," said Adrian. "Add more plasticine," said Robert. They experimented with different lengths until they were satisfied with the beat. "How shall we count?" I asked. "Number of swings in half a minute," said Mervyn. I gave him my watch with a second hand and he was soon timing confidently from any starting point of the second hand. "It swings much faster when we shorten the string," they commented as they continued their experiments with different lengths, "but it doesn't change when we alter the weight." These experiments were rough and ready and needed a careful follow-up with a good point of suspension, but I want to stress that <u>all</u> the thinking had been done by the boys themselves and the suggestions came from them and not from me. "The sense of personal discovery influenced the intensity of their experience and vividness of their memory."[1] We were so engrossed that we forgot all about the purpose of the timing![2]

The initial impulse needs to arise from some experience in which the children are interested. For example, the directed question to find the length and breadth of a room may seem to be of little interest compared with the open-ended question "Do you think this room is twice as long as it is wide? In how many ways can you find out without actually measuring?" which Miss Biggs reported to have aroused the enthusiasm of some nine-year olds.

Discovery learning could be achieved in arithmetic. Here are some topics which will be of interest to children and which with the right questions could give children a sense of achievement and of discovery:

1H.M.S.O.: Children and their Primary Schools, London, H.M.S.O. 1967, para. 549.

2Commonwealth Secretariat: Mathematics in Commonwealth Schools, London, Commonwealth Secretariat, 1969, p.38.

(i) Matching sets (one-one correspondence).

(ii) Counting, matching a number name to each object.

(iii) Cardinal numbers, numbers in sequence, the number line.

(iv) Measuring; length, weight, capacity, time, area, using simple instruments.

(v) Operations of addition, subtraction, multiplication and division with number, length, weight, capacity, money, time.

(vi) Number relationships and the basic laws - commutative, associative and distributive - preceding written calculations. Observation and discovery of patterns and relationships should come before memorisation. For efficient calculation, some memorisation is desirable but this should be reduced to a mimimum.

(vii) Following the knowledge of number and number relationships, children should be given the experience which would encourage them to discover their own methods for long multiplication and division.[1]

Mathematical symbols should be introduced only <u>after</u> varied experience. The steps should be

(i) active experience by the child.

(ii) discussion with other children and with the teacher.

(iii) recording in the child's own words.

(iv) introduction of mathematical symbols.

(v) written calculations devised by the child himself and refined with guidance from the teacher.

(vi) practice as necessary.

In considering mathematical content in the primary school, the school environment and the teacher's mathematical background are two important aspects. There are topics that arise in the classroom and the school premises and from the children's experience in the home, the shops, the streets and the farms according to the environment. Examples are

(i) statistics, collecting data, and expressing them in words and by graphs, reading and interpreting them.

(ii) shapes, common three-dimensional bodies such as cubes, cuboids, cylinders, spheres, prisms, pyramids, polyhedra, plane surfaces such as squares, rectangles, polygons, triangles, circles, etc. and the use of graph paper.

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[1]For a view on long multiplication and division, see p. 16.

(iii) symmetry, in nature as seen in the environment (fruits, leaves, flowers, animals), man-made as illustrated in patterns of various kinds and in many materials including paper, cloth, wood, etc.

(iv) similarity, of solid and plane figures.

(v) limits, small and large, leading to some idea of infinity.

(vi) transformation of translation, rotation, reflection, enlargement.

When children begin to organise their basic number facts and later their knowledge of fractions, the introduction of structural material[1] and devices like the number line help the children to summarise and gain further insight into their experiences. However, children should first have extensive experience of the environment, handle and deal with numbers in practical situations before using commercial structural material.

1See A.T.M.: Mathematics Teaching, No. 24, A.T.M., Autumn 1963.

Suggestions for Action

1. Mathematics in the primary school should be seen as a new approach in both method and content. The teacher should therefore familiarise himself with both aspects through in-service courses and by working through one or more teachers' and source books in primary school mathematics. He should himself do the kind of mathematics to which his children would be exposed.

2. The teacher should acquire practice in planning his lesson, supplying the necessary materials and providing suitable experience and then asking the right questions in the directed or open-ended form as the situation demands.

3. In communities where most children have no more than primary education, the teacher should anticipate the need for social mathematics in adult life. This implies doing more arithmetic topics but the method should consistently lead the children to find out for themselves.

4. The new approach favours group work and individualised teaching. The teacher should therefore keep in view the immediate and future needs of the children who go on to a secondary school or who have to take a selection examination for admission to a secondary school.

5. The teacher and the children should cultivate an eye for mathematics in their environment and the habit of collecting materials for use in mathematics lessons. Here is a list of useful materials, which can be extended or modified according to the environment:

(a) Sorting, matching and counting materials, structural materials:

Seeds, stones, shells, small plastic toys, cubes, rods (length 1 to 10 units), number track, a number line pinned on the wall or marked on the ground in a variety of relevant units, beads on a taut horizontal string.

(b) Shapes:

Boxes, round tins and other containers; leaves, shells, fruits, seeds, flowers, inch and centimetre cubes, beads, balls, globe, mirrors.

(c) Measurement:

Length; canes or softwood, string or fibre, homemade trundle wheel (yard or metre in circumference). Weight; balance scales, springs, coiled wire (hair rollers, extension springs, tendrils of climbing plants). Lever; straight stick or rod to suspend, or to balance on a small wedge of wood, metal washers. Capacity: pots, gourds, coconut shells, pails, tins. Time; string and bob for pendulum. Area; various materials to cover a plane (or curved) surface or a model, e.g. textiles, newsprint, leaves, seeds, squares, identical triangles, etc., mats, geo-boards or peg boards made of softwood with pins or pegs. Rotation; compass, geared wheels, protractor, home-made clinometer.

(d) Constructional Materials:

Commercially produced strips and bolts or drinking straws, building blocks of various shapes and sizes (off-cuts), home made level.

(e) Materials for Communication and Recording:

Squared paper (in inch and centimetre units), coloured paper and card, newsprint or other cheap paper, paints or dyes, brushes, coloured pencils or felt-tipped pens, abacus and rings of dough or paper beads, scissors.

6. A balance should be kept between practical work and abstractions. Many children enjoy practical activities and mathematical games and others are fascinated by number patterns and visual representations in the form of tables, charts and graphs. It is desirable that activities lead to the abstraction of mathematical ideas. Here are some constructional and other activities which give varied opportunities for investigation:

(a) building structures with unit cubes and unit cuboids.

(b) making patterns with unit squares, unit rectangles, unit triangles etc.

(c) making frameworks with split bamboo canes, pawpaw trunks, drinking straws or meccano, making cages from palm piths, baskets and coops from palm leaves.

(d) model-making in card and other materials: sheds and buildings as well as the regular solids (tetrahedron, cube, etc.).

(e) Interesting shapes such as bandstands, drums, balls, ice cream cones, the scoop for serving ice cream; pyramids could be made in the classroom and their properties investigated.

(f) making or investigating toys, for example kites, tops and hoops, doll-dressing. Mechanical toys and devices are not easily available in all countries, but some could be home-made, for example model aeroplanes, wheeled toys and an inclined plane (for experiment), go-carts, slings, geo-boards (for discovering relations of areas of different shapes).

(g) Games such as cricket, football, basket ball and tennis can make children aware of the path of a moving object.

(h) Paper folding, experiments with paint and mirrors lead to investigations of symmetry and enlargement (similarity). Children sometimes invent "coordinates" for themselves when they describe the position of buried treasure on an imaginary island they have drawn. When trying to find the treasure they realise the value of order in the pairs of distances used to fix the position of the treasure and ordered number pairs are then accepted. Coordinates can also be used in symmetrical reflections and other transformations.

The following are the respective mathematical ideas which can be derived from the preceding activities.

(a) The properties of various two-dimensional shapes, from making and handling frameworks with 3, 4, 5, etc. sides, using first equal then unequal strips; the rigidity of the triangle. Sequences of numbers from a series of frameworks of regular polygons made rigid by longer struts.

(b) Ideas of angles from a child's own observations of the hands of a clock and of other changes of direction. Angle properties of parallel lines and angle sum of a triangle from patterns made in paper folding. Rotation and gear ratio from cog wheels of different sizes.

(c) Mathematical similarity (enlargement of scale) in three dimensions, by building a sequence of cubes from unit cubes. Paths on the globe and on the earth. Similarity in two dimensions by building a sequence of squares from unit squares (and other shapes). Maps. Recognition that some shapes (e.g. cubes, spheres, squares), are always similar.

(d) Volume of cubes, spheres, cylinders, cones and pyramids from practical experience, for example, comparison of the weights of clay used to make the objects, or of the water contained by or displaced by them.

(e) Area: approach through irregular shapes, for example comparing the area of two leaves; counting the number of yam heaps covering each of two fields. Many activities lead to the understanding and application of area, such as gardening, weaving mats, doll dressing, making pictures with scraps of fabric. The making of nets of solids gives a useful link between two and three dimensions.

(f) The speeds of various moving objects compared with the child's own speeds.

(g) Transformation: the making of decorative patterns leads to an awareness of the various movements of a unit shape required to make different types of pattern: translation along a line, rotation about a point, reflection. Correspondences of points, and relations between lines and angles, can be investigated. The basic idea of congruence can be firmly established by means of these experiences.

7. There are a number of pupils' books in modern mathematics. The teacher should select one of the series of such books, usually one of those which were written for the locality. Where there is no local series, the choice should be from those which require little adaptation. It is desirable, however, to have a local series which is based on the children's environment, experience and interest.

8. When the medium of instruction is a mother tongue which does not contain words for expressing certain mathematical experiences and concepts, extra care is needed in finding suitable words and expressions in the mother tongue for describing mathematical experiences and concepts. The teacher should identify and recognise the difficulties which arise from cultural differences with a view to overcoming the difficulties. In many such countries, efforts are being made to find appropriate words and expressions in the mother tongue.

9. In countries where the schools cannot afford commercial structural materials such as Stern, Dienes', Cuisenaire, Avon, Structa, Unifix,[1] which are apparatuses from which to make mathematical abstractions, the teacher should devise or improvise aids which are made with local materials. What is important is that such aids and the environment should provide sufficient experience for the children in learning mathematics.

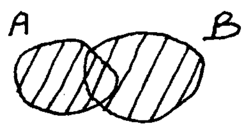
10. The algebra of sets is a common feature in many primary school mathematics books and it makes a lively start, leading on to sorting out, matching, one-one correspondence, numbers, operations, symbols, co-ordinates, graphs. The teacher should be thoroughly conversant with the language, visual representation by Venn diagrams and the symbols, which should be introduced after the children have grasped the concepts. The more usual symbols are:

$\in$	belongs to or is a member of the set
$\subset$	is contained in
$\exists$	there exists
$\cup$	union, sometimes read 'cup'
$\cap$	intersection, sometimes read 'cap'
$\{a, b, c\}$	the set of the elements a, b, c.
$\mathcal{U}$	the universal set or the universe
$\emptyset$	the empty set
A'	the complement of the set A
$\longrightarrow$	tends to
$>$	greater than

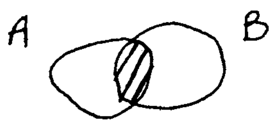
[1]These apparatuses are described in detail respectively in Mathematics Teaching Nos. 16, 17, 18, 19, 20 and 21.

$<$	less than
$=$	equal to
$\simeq$	approximately equal to or is equivalent to the set.

Venn Diagrams - Examples



shaded part is $A \cup B$



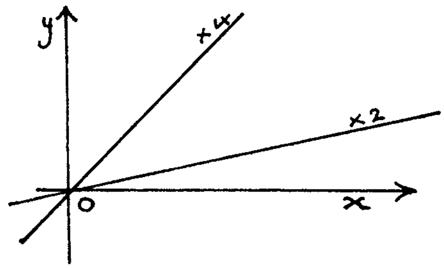
shaded part is $A \cap B$

11. Graphs are a fascinating topic with many applications in number patterns and in daily life. Examples should include simple statistical representations by pictographs, pie graphs, block graphs, column graphs, line graphs.

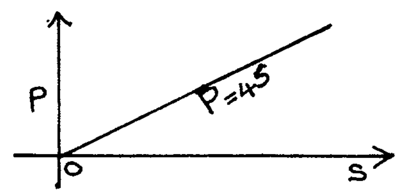
Certain number patterns may be represented by continuous line graphs:

(a) Straight lines

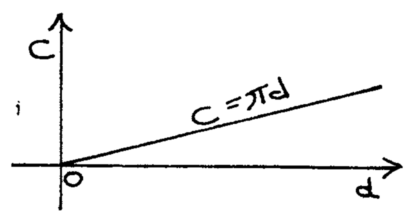
(i) The pattern of the graphs of multiplication tables of 2, 3, 4, etc. as straight lines of increasing steepness. Note the extension to multiplication tables of -2, -3, -4.



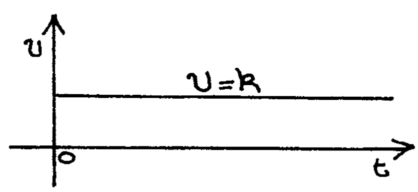
(ii) The perimeter/side graph for squares.



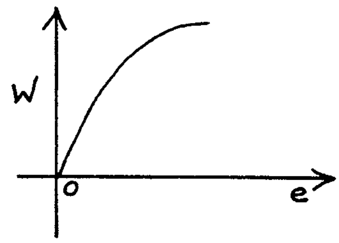
(iii) The circumference/diameter graph for circles.



(iv) A graph showing constant speed..



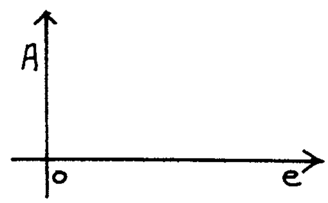
(v) The extension of a spring as weights are added to one end, provided a certain limit of weight is not exceeded.



Children sometimes predict that the graph of the multiplication tables or perimeters of squares will be a straight line because of the equal differences.

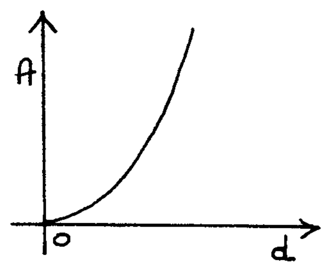
Edge of Squares	1	2	3	4	5
Perimeter	4	8	12	16	20
Differences		4	4	4	4

(b) Graphs of squares and cubes



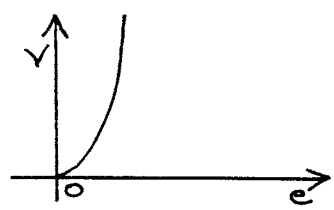
(i) The area/edge graph for the squares.

(ii) The area/diameter (or radius) relation for circles found approximately by counting squares. The number pattern of the area of square is soon recognised.



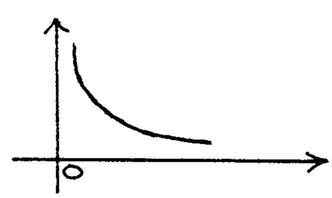
Edge of Squares	0	1	2	3	4	5
Area of Squares	0	1	4	9	16	25
Differences		1	3	5	7	9

Some children realise that because these differences are not equal, the graph will not be a straight line. Because the differences increase and form the odd number pattern they expect a rising curve.



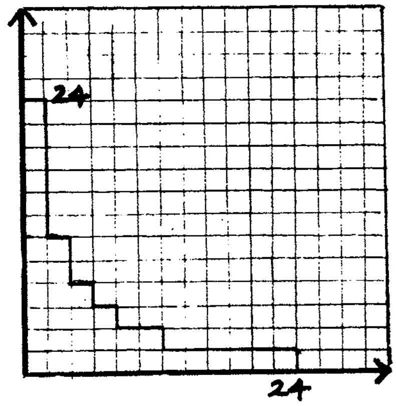
(iii) The volume of cubes and of spheres will be investigated and will lead in a similar way to a curve.

(c) Constant product curve



(i) Examples of the constant product pairs have already arisen, e.g. in the multiplication square.

(ii) Dimensions of rectangles made with 24 squares. The complete set of rectangles with integral dimensions can be cut out and put in order: 1 by 24, 2 by 12, 3 by 8, 24 by 1. When children are asked to arrange these in order taking up the least possible space, they sometimes overlap them and then recognise the similarity to the pattern in the multiplication square.



Ordered pairs, (1,24), (2,12), ..., (24,1), with relation lb = 24 give a continuous curve.

12. As a primary school teacher usually handles all the subjects in his class, he should make use of the links of mathematics with other subjects of the curriculum, especially with science, social studies, arts and crafts. A class may be given a project which demands the knowledge of mathematics, science and social studies. In the rural areas, agricultural situations should form the basis of the problems. Interesting mathematical problems could arise from poultry farms, fishponds, pig, cocoa, citrus, and rice farms, sale of produce, banana, coconut and oil palm farm practices.

13. The teacher should be involved in the making of the syllabus for his class. He should select those topics which arouse the interest of the children, assist them in their exploration and discovery and enrich their mathematical experience. The degree of computational skill required will depend on the society and its environment. Where calculating machines, ready reckoners and the slide rule are in common use, long multiplication and division skill is no longer necessary. Here is a suggested curriculum from which a syllabus could be made. The teacher should feel free to re-arrange the topics in the best way dictated by his experience of the children and their problems and language and in relation to the other subjects of the school curriculum.

1. Mathematical concepts of number and space drawn from concrete familiar objects within the experience of the children.

2. Sets and sub-sets, leading to union, intersection and complements of sets and sub-sets.

3. Numerals in the mother tongue and English.

4. Symbols, the denary number system and the digits.

5. Counting numbers, other kinds of number, by actual counting and the use of aids.

6. Number bases - 2, 3, 4, 7, 10, 12, using as illustrations the computer, market days, the week, the denary number system and the calendar year.

7. The operations of addition, subtraction, multiplication and division with reservation as to the need for teaching long multiplication and division skill.

8. The basic laws of arithmetic (and algebra) - commutative, associative and distributive.

9. Measures of money, time, weight, length, area, volume, by actual measurements and handling of objects in and around the classroom.

10. Other measures, e.g. temperature, rainfall, pressure, density, leading to statistical data.

11. Elementary graphs and charts.

12. Number patterns, tabular and graphic representations.

13. Transformations of translation, rotation, reflection, enlargement.

14. Direction, angles, simple plane figures, shadows and sun-dials.

15. Shapes of solid and plane figures, simple calculation of areas and volumes.

16. Measures in the mother tongue units and the metric system and the equivalents to one another.

17. Practical applications and illustrations, including games which involve mathematical ideas and patterns, such as ludo, snakes and ladders, dominoes, draughts, chess and local games.

Selected Teachers' Books

1. Association of Teachers of Mathematics: Notes on Mathematics in Primary Schools, London, Cambridge University Press, 1967.

2. A.T.C.D.E.: Primary Mathematics for Schools and Training Colleges, London, A.T.C.D.E., 1963.

3. Biggs, E.E. and Maclean, J.R.: Freedom to Learn: Ontario, Addison-Wesley (Canada) Ltd., 1969.

4. British Council: New Approaches to Teaching Mathematics (a book list), London, British Council, 1970.

5. Bryan Thwaites: On Teaching Mathematics, London, Pergamon Press, 1961, 116p.

6. Cockcroft, W.H.: Your Child and Mathematics London, W. & R. Chambers and John Murray, 1968, 28p.

7. Dienes, Z.P.: Mathematics in the Primary School, London, Macmillan, 1964.

8. H.M.S.O.: Children and their Primary Schools, London, H.M.S.O. Volume 1, 1967, 555p.

9. H.M.S.O.: Mathematics in Primary Schools, London, H.M.S.O., 1966, 165p.

10. Jay D. Weaver and Charles T. Wolf: Modern Mathematics for Elementary Teachers, Scranton (Pennsylvania), International Textbook Company, 1968, 274p.

11. Kennedy, J.: Understanding Sets, London, Thomas Nelson and Sons Ltd., 1967.

12. Lackie, L.: Understanding Shapes and Solids, London, Thomas Nelson and Sons Ltd., 1967.

13. Land, F.W. (Ed.): New Approaches to Mathematics Teaching, London, Macmillan, 1963.

14. Lee, D.M.: Background to Mathematical Development, London, Oldbourne, 1962.

15. Lucienne,Felix: Modern Mathematics and the Teacher, Cambridge, Cambridge University Press, 1966, 128p.

16. Meredydd,G. Hughes: Modernising School Mathematics, London, G. Bell & Sons Ltd., 1962, 43p.

17. Polya, G.: How to Solve It, New York, Doubleday, 1957, 253p.

18. Rodda, G.M.: Understanding Number, London, Thomas Nelson and Sons Ltd., 1967.

19. Rodda, G.W.: Understanding Graphs and Statistics, London, Thomas Nelson and Sons Ltd., 1968.

20. Schools Council: Mathematics in Primary Schools, London, H.M.S.O., 1966.

21. Scopes, P.G.: Mathematics for Primary School Teachers, London, Longmans, 1969, 120p.

Periodicals

1. Association of Teachers of Mathematics: Mathematics Teaching (the quarterly bulletin of the Association of Teachers of Mathematics), subscriptions to Hon. Treasurer Ian Harris, 122 North Road, Dartford, Kent, England.

2. The Mathematical Association: The Mathematical Gazette (the journal of the Mathematical Association), London, G. Bell and Sons Ltd., Portugal Street, London, W.C.1.

Notes:

1. Both periodicals carry a wide range of reviews and advertisements of mathematics books.

2. Back numbers contain illuminating articles on primary school mathematics.

Selected Pupils' Textbooks

The choice of pupils' books depends on what is available locally or can be obtained with minimum difficulty and on the extent to which the teacher depends on pupils' books. The ideal books are those specially written with the children and their environment in view. Some of them are in mother tongues other than English. The next best are those books which require little adaptation on account of their general appeal to children and their interests.

4. MATHEMATICS IN SECONDARY SCHOOLS[1]

A consideration of mathematics in secondary schools might begin with three fundamental types of questions. The first type revolves around what to teach and includes such questions as, "In an age of calculating machines need children know multiplication tables? Should we teach manipulation in algebra? If so, how much? Are logarithms obsolete? Should we teach trigonometry at all? If so, why? What parts of recent mathematics ought to come into the school syllabus? What should go out to make room for new topics? More fundamental, what considerations should determine our choice of syllabus?" The second type deals with the equally and perhaps even more important problem of how we teach. "Should mathematics fuse with science and the study of the environment, or should it be taught as abstractly as possible? Should it be formal or informal, rigorous or intuitive? In what proportions should discovery and 'telling' be mixed?" The third type concentrates on the objectives of mathematical education in secondary schools and asks how they can be achieved. This type asks why we teach what we teach in the way we teach it? How do we achieve our objectives?

The answers to these questions would vary from country to country and might vary from place to place in the same country, depending on the kind and quality of the teachers, the children and their interests, present and future needs, the aims and aspirations of the nation, the financial resources of the country and the general and special attitudes of the people to mathematical and other forms of education. Mathematical education in secondary schools should be based on the two strands of objectives which are already outlined[2] and which should be liberally interpreted to cover the various utilitarian aspects and the higher ideals of education. A minimum programme should achieve the following objectives:

(1) The pupil should enjoy mathematics and not be afraid of thinking about it.

(2) He should be able to grasp mathematical results informally, pictorially or in terms of a concrete situation.

(3) He should have worked with mathematics in connection with simple scientific laws and with the environment generally.

(4) In this way he should have acquired as much knowledge and understanding of arithmetic, algebra, geometry (and perhaps other mathematical subjects) as possible.

In many countries, secondary education consists of five years of broadly based education and two years of narrow but more specialised forms of education, usually referred to as Sixth Form. The first three years are devoted to an even broader educational base. In most countries, the Sixth Form is the preparatory class for University work. Pupils who select mathematics for their specialisation are in the range of the able and gifted children and their teachers are usually in the run of the better qualified teachers of mathematics. The mathematical education of the bigger group, that is Classes (Forms) One to Five needs special attention so that the

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[1] Adapted partly from the lead paper by Professor W.W. Sawyer.

[2] See p.3.

approach already started in the primary school may continue and that adequate mathematics may be provided in pursuance of the objectives of mathematical education at that level.

In planning a secondary school curriculum in mathematics, knowledge of the primary school background as to content and approach is important. The selection entrance examinations (where they are held) should concern themselves not only with the acquisition of skills but also with the development of understanding and attitudes. It is desirable that teachers should have considerable freedom in deciding syllabus content, although it would be unrealistic to think that all teachers are yet ready for this responsibility. The setting up of Syllabus Committees in countries where they do not exist should be encouraged. On these committees teachers should play a dominant role, but personnel from industry and Government should be included. In cases where external examinations are set, the examining bodies should be guided by local requests.

A school curriculum should take cognisance of the school organisation and facilities. Timetables should be arranged to permit 'setting' in mathematics. Under this arrangement a group of classes in a particular year have mathematics simultaneously, the students being assigned to classes according to ability in mathematics. This arrangement allows a student to transfer at any time during the year to a class which accommodates his ability. Classroom furniture should be chosen so as to facilitate group working. Flat-top tables are preferred to sloping ones. Where possible mathematics teachers should have individual classrooms to which classes may go in turn. Failing this, there should be a mathematics room or a laboratory suitably equipped. This moderate requirement may be beyond the resources of some schools but it should be kept in view as a minimum requirement.

All children in the first three years of their secondary education should pursue courses with a common core. Those children who are more able in mathematics should be given the opportunity to study topics in greater depth and to consider additional topics. At the other end of this ability range, topics should be treated in a more informal manner, but at the same time such children should not be deprived of an enrichment programme. The following principles for the design of a mathematics course are recommended:

(i) The establishment of a natural link with primary school teaching which should make the transition from primary to secondary school as smooth as possible.

(ii) In some situations a practical approach involving student activity would be a useful teaching technique to use. Teachers of mathematics in secondary schools should be familiar with and take an interest in primary school methods and encourage the continuance of these where appropriate.

(iii) In secondary schools there should be a shift of emphasis to systematising and formalising, as well as expanding the concepts and experience of the primary school.

(iv) To make the course relevant to the community and its mathematical needs as far as they can be identified.

(v) The school curriculum should be integrated so that the contribution of a subject to each of its related subjects can be used to the best advantage e.g. the mathematical concepts necessary to physics should be co-ordinated with the physics programme and, on the other hand, aspects of physics which could serve to illuminate the need for mathematical concepts should be co-ordinated with the mathematics programme.

(vi) Proof or some other form of justification should be given for all formulae, algorithms and theorems considered.

(vii) In cases where the child's formal education is to terminate after 3 years, he must leave school not only with an acceptable degree of numeracy (i.e. insight into mathematical thinking in general and into all basic arithmetical techniques) but also with that flexibility of mind which would enable him to apply himself to problems he will meet as a member of his community.

In considering the needs of pupils in their first three years of secondary education no attempt is made to lay down a syllabus or to suggest the order in which topics should be taught. The depth of treatment of these items, and the way in which they would be introduced would vary from country to country and indeed from class to class or even pupil to pupil. For example, although all pupils would be expected to be able to solve linear equations, quadratic equations would be included only when there was a clear need. The following topics are suggested as those which should be encountered by all students:

Set

The idea of a set, set language and notation; union, intersection, complement, empty (null) set, universal set, subset; illustrations by Venn diagrams; the basic laws of set.

Number

Number line, extension to negative numbers, rationals, irrationals, and reals.
Place value, scientific notation (e.g. 2.38×10^{-6}).
Computational aids (e.g. the slide rule).
Percentage, ratio, approximation and error estimation.

Algebra

Variable, functions.
Algebraic expressions and operations.
Relations, equivalence relations.
Inequalities, identities, formulae.
Equations, solution of equations.
Graphical representation.

Geometry

Area and volume, the theorem of Pythagoras.
Similarity, angle measure and angle properties.
Elementary trigonometry.
Plans and elevations.
Co-ordinate geometry.
Transformations of translation, rotation, reflection, enlargement.

Identity elements and inverse elements should be studied in geometrical as well as in numerical and algebraic contexts.

Statistics and Probability

It is suggested that consideration of probability should spring from the pupil's own experiments, and that the statistics studied should be confined to representing data by means of graphs or tabulation, the interpretation of data exhibited in these forms, and the use of the measures mean, mode and median.

Terms such as equivalence relation are not expected to be used by pupils, or necessarily to be defined explicitly, but nevertheless the ideas should permeate the teaching. For example, children should realise the similarity between 'is equal to', 'is parallel to' and 'was born in the same year as'.

It must be stressed that this skeleton plan must be supported by a full enrichment programme appropriate to the pupils concerned. For example, matrices might be studied not only for their own sake but also to increase the pupil's understanding of such items as operations, equations, inverse elements and co-ordinate geometry; the study of transformation or vector geometry would, amongst other things, greatly increase a child's experience of spatial relations. However, the choice of such topics should be left to individual countries or teachers.

When considering the syllabus for the fourth and fifth years, it is desirable to bear in mind the fact that for many students this would mark the end of their formal education in mathematics. For such students topics with social and economic relevance should be given particular consideration. The following topics are thought to be appropriate:

(a) Linear programming.

(b) Statistics up to sample theory and significance (including projects of an experimental nature).

(c) Co-ordinate geometry and vectors.

(d) Operatives in systems other than the real numbers e.g. matrices, geometrical transformations.

(e) Computing, flow charts, programming in some subsets of high level language, Computer appreciation.

(f) Social arithmetic e.g. income-tax, hire-purchase, etc.

The choice of topics at the Sixth Form level should be sufficiently wide to satisfy the needs of

(i) those whose formal education in mathematics will terminate at the end of this stage;

(ii) future social and biological scientists;

(iii) future physical scientists and engineers;

(iv) future specialists in mathematics.

Some of the principles which should guide the choice of content are:

(i) the need to systematise parts of mathematics, for instance, co-ordinate geometry, properties of polynomials, elementary number theory;

(ii) the need for refining the ideas of proof and the use of axiomatic processes where possible. For example, elementary abstract algebra can be used to justify rigorously results presented informally in the earlier grades. On the other hand, the introduction to the calculus should be based on an intuitive approach to limits;

(iii) the need to make mathematical models of physical situations: i.e. abstracting from a physical situation by symbolising quantities and relations and arriving at mathematical conclusions in the model which can be interpreted in terms of the physical situation. Such illustrations are to be found in mechanics and probability;

(iv) the need to develop facility in the use of mathematical methods.

Patterns of collaboration with industry and technology depend on each country, its resources and circumstances. Collaboration might take the form of getting schools, industry and technology to define aims and scope of the mathematics curriculum for the country, or of representatives of industry and technology participating in teaching in the schools and in the process of examinations. There are examples of national forums for curriculum development.

Modern mathematics in the secondary school should reflect in approach and content the objectives of a good mathematical education, bearing in mind the needs of the country and the power of mathematics not only as an intellectual exercise but also as an instrument for the general education of the person.

Suggestions for Action

In many countries, a secondary school mathematics teacher is a specialist teacher. He is usually the holder of a degree in mathematics or in a combination of two or three subjects of which mathematics is one. He may in addition have a teaching qualification. If he is not a graduate, he would have had two or three years of teacher training and further mathematical education beyond the level represented by the ordinary level of the General Certificate of Education in mathematics. In countries where teachers with lower qualifications teach mathematics in a secondary school, the situation is regarded as an emergency requiring urgent attention. The Sixth Form mathematics teacher usually holds an honours degree in mathematics or its equivalent.

It is essential for the teacher to bring himself up-to-date in the aims, content and method of teaching modern mathematics,[1] through reading,

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[1] A detailed account of the terms "Traditional", "Modern" and "New" Mathematics is contained in the Unesco Mathematics Project for the Arab States, published by Unesco in January 1969.

attentance at conferences, seminars and in-service courses. A practical step for the teacher is to subscribe to a mathematics professional journal like the Mathematical Gazette which carries contemporary articles, reviews and advertisements of mathematics books. The secondary school teacher should acquaint himself with the primary school mathematics curriculum, its objectives, content and approach.

The following suggestions are made with the above basic assumption of the teacher's competence and continued up-dating:

(1) Links between primary and secondary schools should be established to ensure that transition is as smooth as possible.

(2) Teachers should be given considerable freedom in deciding syllabus content. It is often desirable to set up a syllabus committee on which teachers play a major rôle and both industry and Government are represented.

(3) Secondary courses should be relevant to the various mathematical needs of the community in which they operate.

(4) In the first three years of secondary schooling, mathematics courses should have a common core; thereafter they should be varied to suit the needs of particular groups.

(5) Mathematics and other subjects of the curriculum should be inter-related so that the contribution of each subject can be used to the best advantage.

(6) Timetables should be planned to allow simultaneous teaching of different ability groups within a particular year.

(7) To facilitate group working and practical classroom activities, suitable furniture should be chosen.

(8) It is essential to provide a mathematics room and, where possible, a suitably equipped mathematics laboratory.

(9) In most educational systems of the English-speaking countries, teachers have freedom to try new methods and influence the curriculum and the syllabus. Advantage of that freedom should be taken to to try group work and individualised teaching, identify the able and the weak pupils and help them develop according to their ability and needs and to select those topics which assist understanding and promote skill.

(10) A number of examining bodies are disposed to co-operate with teachers and base their examinations on representative curricula. They welcome criticisms of the examinations and their relevance to the school curricula. That freedom should be exploited to ensure that examinations are based on the curricula and not the other way round.

(11) Many school certificate mathematics examinations consist of two sections, one on the core topics and the other on a wide range of options. Teachers should use the advantage of the pattern to select the topics which best fulfil their objectives of mathematical education.

Selected Teachers' Books

1. Austin, J.L.: The Foundations of Arithmetic, Oxford, Basil Blackwell, 1950.

2. Board of Education: Curriculum and Examinations in Secondary Schools, London, H.M.S.O., 1946.

3. Bruce, George: Secondary School Examinations Facts and Commentary, London, Pergamon, 1969.

4. French, P.: Number Systems, London, The House of Grant Ltd., 1964.

5. Hardy, G.H.: A Mathematician's Apology, Cambridge, Cambridge University Press, 1940.

6. Incorporated Association of Assistant Masters in Secondary Schools: The Teaching of Mathematics, Cambridge, Cambridge University Press, 1960.

7. Land, F. W. (ed.): New Approaches to Mathematics Teaching, London, Macmillan, 1963.

8. Meredydd,G. Hughes: Modernising School Mathematics, London, G. Bell & Sons Ltd., 1962.

9. Ministry of Education: Teaching Mathematics in Secondary Schools, London, H.M.S.O. 1958.

10. Ministry of Education: The Road to the Sixth Form, London, H.M.S.O., 1951.

11. Ministry of Education: 15 to 18, London, H.M.S.O. 1959.

12. Nunn, T.P.: The Teaching of Algebra, London, Longmans, 1941.

13. Peel, E.A.: The Pupil's Thinking, London, Oldbourne, 1967.

14. Pedoe, Dan: The Gentle Art of Mathematics, London, Pelican, 1965.

15. Polya, G.: How to Solve It, New York, Doubleday, 1957.

16. Reichmann, W.J.: The Fascination of Numbers, London, University Paperbacks, Methuen, 1965.

17. Russell, Bertrand: Introduction to Mathematical Philosophy, London, George Allen and Unwin Ltd., 1950.

18. Sawyer, W.W.: A Concrete Approach to Abstract Algebra, London, W.H. Freeman and Co. 1959.

19. Sawyer, W.W.: Prelude to Mathematics, Harmondsworth, Penguin, 1955.

20. Sawyer, W.W.: Mathematician's Delight, Harmondsworth, Penguin, 1959.

21. Sawyer, W.W.: Vision in Elementary Mathematics, Harmondsworth, Penguin, 1964.

22. Sawyer, W.W.: A Path to Modern Mathematics, Harmondsworth, Penguin, 1966.

23. Secondary School Examinations Council: The Certificate of Secondary Education - some suggestions for teachers and examiners. London, H.M.S.O., 1963.

24. Secondary School Examinations Council: The Certificate of Secondary Education - An Introduction to some Techniques of Examining, London, H.M.S.O., 1964.

25. The Mathematical Association: The Mathematical Gazette (the journal of the Mathematical Association), London, G. Bell and Sons, Ltd.

26. The Mathematical Association: A Second Report on the Teaching of Arithmetic in Schools, London, G. Bell and Sons Ltd., 1964.

27. The Mathematical Association: The Teaching of Arithmetic in Schools, London, G. Bell and Sons Ltd., 1946.

28. The Cambridge University Mathematics Society: Eureka (the Journal of the Archimedeans), Eureka c/o The Arts School, Benet Street, Cambridge, England.

29. The National Foundation for Educational Research in England and Wales: Educational Research, London, Newnes Educational Publishing Co.

30. The Mathematical Association: A Second Report on the Teaching of Mechanics in Schools, London, G. Bell and Sons Ltd., 1965.

31. The Mathematical Association: Transfer from Primary to Secondary Schools, London, G. Bell and Sons Ltd., 1964.

32. Thwaites, Bryan: On Teaching Mathematics, London, Pergamon Press; 1961.

33. UNESCO (Education Clearing House): The Teaching of Mathematics by G. Mialaret, Paris, Unesco, 1959.

34. University of Southampton: Aspects of Modern Mathematics, Southampton, The University, 1963.

35. Vernon, Philip E.: The Certificate of Secondary Education - An Introduction to Objective-Type Examinations (Examinations Bulletin No. 4), London, H.M.S.O. 1964.

36. Wertheimer, M.: Productive Thinking, New York, Harper, 1945.

Selected Pupils' Textbooks

There is a wide range of secondary school mathematics books written in the developed countries and distributed widely in developed and developing countries. The practice is based partly on the assumption that mathematics is ultimately culture-free. However, there is a welcome trend of developing mathematics curricula and syllabuses on a country-basis; authors in the future will write books specifically aimed at each country or region. In this trend, two predominant sources are the School Mathematics Project (S.M.P.) of the United Kingdom and the School Mathematics Study Group (S.M.S.G.) of the United States of America, supported by the Illinois Experimental Programme. Books by the Scottish Mathematics Group (S.M.G.) have appeared in West Africa as have some single-author books.

The School Mathematics Project books are used in many African countries, East and West. They are being adapted and re-written to suit local regions. Thus there are in East Africa (with Makerere as the base) the School Mathematics Project for East Africa and, in Ghana, the Joint Schools Project (J.S.P.) which produce books based on the S.M.P. material.

The ideas of the S.M.S.G. and the Illinois Experimental Programme came through the Educational Services Incorporated (E.S.I.) and now the Education Development Center (E.D.C) which organised the African Mathematics Programme. The books produced under this programme are known as Entebbe Mathematics, after Entebbe, Uganda, where most of the workshops for preparing the material were held. In the curriculum development in the developing countries, there will emerge new material incorporating the best from both sources.

A number of publishers have modern mathematics books, which are worth 'inspecting'. Whatever textbooks are adopted, secondary school pupils should be encouraged to read a wide range of mathematics books, including biographies, histories, pastimes. As a minimum reading, Bell, Hardy and Sawyer are recommended. Here is a short general reading list.:

1. Austin, J.L.: The Foundations of Arithmetic, Oxford, Basil Blackwell, 1950.

2. Bell, E.T.: Men of Mathematics, Vols. 1 & 2, Harmondsworth, Penguin, 1937.

3. Hardy, G.H.: A Mathematician's Apology, Cambridge, Cambridge University Press, 1940.

4. Kennedy, J.: Understanding Sets, London, Thomas Nelson and Sons Ltd., 1967.

5. Lackie, L.: Understanding Shapes and Solids, London, Thomas Nelson and Sons Ltd., 1967.

6. Pedoe, Dan.: The Gentle Art of Mathematics, London, Pelican, 1965.

7. Polya, G.: How to Solve It, New York, Doubleday, 1957.

8. Reichmann, W.J. : The Fascination of Numbers, London, University Paperbacks, Methuen, 1965.

9. Rodda, G.W.: Understanding Number, London, Thomas Nelson and Sons Ltd., 1967

10. Rodda, G.W.: Understanding Graphs and Statistics, London, Thomas Nelson and Sons Ltd., 1968.

11. Russell, Bertrand: Introduction to Mathematical Philosophy, London, George Allen and Unwin Ltd., 1950.

12. Sawyer, W.W. A Path to Modern Mathematics, Harmondsworth, Penguin, 1966.

13. Sawyer, W.W.: Mathematician's Delight, Harmondsworth, Penguin, 1959.

14. Sawyer, W. W.: Prelude to Mathematics, Harmondsworth, Penguin, 1955.

15. Sawyer, W.W.: Vision in Elementary Mathematics, Harmondsworth, Penguin, 1964.

The following book lists are useful in selecting books for a School Library.

1. The British Council: New Approaches to Teaching Mathematics, London, 1969.

2. The Mathematical Association: School Library Mathematics List, London, G. Bell and Sons Ltd., 1966.

3. University of Southampton: Aspects of Modern Mathematics, Southampton, the University, 1963.

The School Mathematics Project books are published by Cambridge University Press, Cambridge and the Entebbe Mathematics books by the Education Development Center, Newton, Massachusetts, U.S.A.

5. ASSESSMENT OF CHILDREN'S PROGRESS AND EVALUATION OF PROGRAMMES[1]

Introduction

A few years ago it was apparently easy to examine candidates in mathematics and it seemed comparatively simple to evaluate a mathematics curriculum. At the primary school level the content of the teaching was confined to arithmetic and the debate was in terms of a conventional testing of mechanical and problem arithmetic contrasted with objective, multiple choice tests. The situation at the secondary level was equally stable. There was an interesting and valuable move away from examining mathematics in separate compartments by means of papers in algebra, geometry, arithmetic and trigonometry towards papers in mathematics as a whole.

Three developments have taken place recently or are taking place, which make the problem of examination, assessment and evaluation increasingly complex. The first is the development of new ways of examining which place much more emphasis on the assessment of course work and the opinion of the teacher.[2] The second is the development of new ways of learning and teaching which, in a number of countries, tend towards the abolition of rigid streaming by ability and the forming of classes of widely varying capability, particularly in mathematics.[3] Efficient class teaching is almost impossible in such circumstances and the good teacher is obliged to consider the formation of small groups within the larger class, and the adoption of individualised learning by means of pieces of work set for pupils to work on their own. There is support, of course, for group work and individualised teaching as conducive to active ways of learning.[4]

The third trend is the development of new content at both the primary and the secondary levels. At the primary level we are no longer content to teach the old arithmetic; mathematics has taken its place. Sometimes this mathematics takes a quite traditional form, such as simple geometry and trigonometry, or the use of graphical methods; sometimes the change is more radical in terms of sets, transformation geometry, etc. Particularly in the primary schools, the development of new mathematics and new teaching and learning procedures are often combined together in such a way that their effects cannot be separated. In the secondary schools, so far, the change has been, in the main but not entirely, towards new content.

The fourth trend, especially in developing countries, is the phenomenal rise in the numbers of candidates. Mathematics in one form or another is a subject of the selection examinations into the secondary school and one of the subjects offered by most candidates in the school certificate examination. This trend is responsible for an increasing automation in the marking of students' examination scripts.

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[1] Adapted from the lead paper by Professor J. Wrigley.

[2] Secondary School Examinations Council: The Certificate of Secondary Education - some suggestions for teachers and examiners, London, H.M.S.O. 1963.

[3] cf. Comprehensive Schools.

[4] E.A. Peel: The Psychological Basis of Education, London, Oliver and Boyd, 1964.

All these developments lead to considerable problems in examination and evaluation. Although the examination of candidates and the evaluation of curricula are often considered as separate problems, a little consideration readily shows that they are inextricably interwoven. An examination first differentiates between the candidates and then fixes a general standard of achievement. If we are examining pupils who have been taught to a new and radically different syllabus, it is comparatively easy to grade the pupils in an order of merit (the first function of an examination), but it is much more difficult to make judgments about standards. Questions of comparability with the old standards arise; sometimes they can hardly be immediately answered. As soon as we make statements about standards we are also evaluating the curriculum as well as the teaching and the pupils. So our present task with regard to examination, assessment and evaluation is at once both more interesting and more difficult than it used to be. And in the immediate future the difficulties are likely to increase rather than be resolved.

Examinations

External examinations have played an important part in the development and progress of many countries. The trouble with most examinations is that we ask of them too much. We expect them to differentiate between candidates, to provide evidence of standards, to act as guides to good teaching, and to provide incentives for both pupils and teachers. We expect them to predict future performance as well as to certify that a candidate has completed satisfactorily a course of recognised study. Add to all this the fact that we examine candidates in their tens of thousands and it is not surprising that we run into difficulties. Ideally, we should separate these various functions, asking ourselves exactly what we aim to do, and design instruments to carry out our aims. But we are rarely able to do this.

A most interesting development in the realm of examining in recent years is the examination for the Certificate of Secondary Education (C.S.E.)[1] in England and Wales. It was set up to provide an examination suitable to the needs of pupils in the secondary schools, who were not in the top ability groups. It was designed roughly for pupils between the 80th and 40th percentile of the ability grouping. Thus it was of a lower standard than the General Certificate of Education (G.C.E.) Ordinary Level and intended to suit average children and above. It was to take the place of a number of external examinations which had been taken by these pupils and which had become rather remote from the needs of the pupils in school. To ensure that the new examination was relevant, teachers were put in control of all the important committees and fourteen C.S.E. Boards were set up throughout the country. The philosophy of C.S.E. is that teachers and examiners should coincide, should often be the same people, that a teacher knows best the capability and calibre of his own pupils but that he needs the help and guidance of external examiners in the final determination of national standards. For many subjects, and for pupils of average ability, it was clear that conventional methods of examination were no longer adequate, so the C.S.E. was designed so that it could be taken in three different ways or modes. Mode I is a conventional external examination, Mode II is an examination set externally on a syllabus designed by the pupils' own school and teacher, Mode III is an internal examination externally moderated. The latter is a means by which an enterprising teacher can examine his own work

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[1] Secondary School Examinations Council: The Certificate of Secondary Education - some suggestions for teachers and examiners, London, H.M.S.O., 1963.

and his own syllabus in the way he thinks best within his own class, though of course he is subject to the checks of an external moderator. The C.S.E. Board has the final responsibility of making sure that work under Mode III is comparable in scope and in standard with work under either Mode 1 or Mode II.

The development of Mode III examinations with the associated trend to continuous assessment and the evaluation of course and project work has been uneven throughout the country and has depended upon the policy of the individual examining boards. Certain subjects of the curriculum such as rural studies, home economics, music seemed to cry out for the imaginative use of Mode III procedures, but a similar development has taken place in the more traditional and academic subjects, and mathematics has been one of these. In particular, those teachers who have been quick to see the value of new content in mathematics and/or the open-ended way of teaching have wanted to experiment with Mode III. The intention of many of these people has been to try to reproduce, albeit at a lower level, the kind of activity of a professional mathematician. This demands project work, essay-type questions, open-ended situations, and a whole new style of teaching, learning and examining. The number of these vanguard teachers is not necessarily large but it is growing. The problems raised for the moderator in such a situation are considerable. How is he to make sure that standards have been maintained, that the subject taught is still recognisably mathematics? Given tact and understanding, a reasonable assessment and evaluation can take place. The teacher must be trusted to assess his own pupils to a large extent especially as regards an order of merit within the class. But the moderator might expect, even with the most esoteric subject matter and the most free and easy learning situations, that on some key aspects of mathematics traditional questions could be asked. It is evident, though, that this kind of compromise does not completely solve the problem of comparability - the trustworthiness of the teacher and the wisdom of the moderator are essential elements in the situation.

The new ideas of teachers and examiners coming together in one person, of ratification of progress rather than external examination, of continuous assessment rather than a once-and-for-all external examination, of the assessment of course and project work are present to a greater or lesser extent in all the three Modes of examination and in all the C.S.E. Boards. A system of internal examinations, continuous assessment, project work, open-book type examinations has not yet been extensively tried out in highly competitive situations. With most of these ways of examining, the interaction between teacher and pupils becomes rather obvious. In examining a dissertation or a long essay the influence of the tutor on the work is considerable and it is sometimes difficult to know how to allow for this to give a 'fair' assessment. The fact that a similar situation exists in an external conventional examination (anyone who has examined large numbers of mathematics scripts at G.C.E. 'A' and 'O' levels will know that he is examining schools and teachers as much as candidates) does not mean that we have a solution to the problem.

The kind of development in examining that has been described is comparatively sophisticated. It takes the risk of bringing in the teacher as an examiner, it admits openly the possibility of bias; it is difficult to accomplish in a highly competitive situation. Such methods will flourish best in a system which provides expanding opportunities for education where the effects of failure can be retrieved at a later date. It is an interesting question as to whether a similar movement is suitable for developing countries. Since these new ways of examining (at their best) lead to more relevant and less artificial teaching and learning perhaps they are all the more essential. But they demand well qualified teachers who know exactly where they are going and what they are doing. The external London G.C.E. (and even degree examinations) set a standard which is widely recognised, but sometimes at the expense of some unreality. Perhaps a compromise is necessary. Part of the examination in mathematics could be a conventional two or three hour written paper; this would take care of the essentials and provide an external yardstick. Part could be an assessment of course work, and open-ended examination - this would encourage real mathematics in the schools.

A development of examinations is the objective-type as distinguished from the conventional type which demands written statement, step by step, leading to the answer. An examination, either the objective-type or the conventional type, is devised to assess the attainment and skill of pupils in a particular subject. All the questions refer to a syllabus defined for the class concerned. An important difference between the objective and the conventional written examination is that the former usually consists of a large number of questions which cover the syllabus more extensively and thoroughly. In the conventional examination, the questions are few and the coverage is not as systematic and extensive. There is therefore a greater chance that the questions included will suit some pupils very well and others less so, in spite of the practice in some papers of allowing candidates to select a few questions from the paper. The issue does not arise in an objective examination where candidates are expected to answer all questions, which cover the whole syllabus.

The objective-type examination has been viewed with suspicion. Some doubt if one-word answers can ever be an adequate form of assessment and think that even if it were efficient the backwash effect on the teaching would be bad. The second objection is true of many external examinations, but the first is not true if the examination is well constructed. Good objective-type examinations can examine high level modes of thinking efficiently. The pressure of examining large numbers of pupils will force us in one of two directions either towards more teacher and school-based assessments or towards more multiple choice tests which can be scored by machine. Many examining boards now use some objective-type in school examinations, such as the selection examinations for admission into secondary schools and the school certificate examinations comparable to the G.C.E. Ordinary level.

The most usual form of the objective-type examination is the multi-choice in which a candidate is set the task of selecting a current answer from a list of alternative answers offered. A later development is the multi-facet form in which a mathematical situation with many facets is presented and the candidate is required to make a decision on each of the facets. In exploiting this multi-facet idea statements were phrased so that some were true and some were false and the candidates were required to decide the truth or falseness of each statement made about each situation. Here are two examples of the multi-facet form of the objective-type examination:

(1) If $a = \frac{1}{2}$, $b = \frac{2}{3}$ and $c = \frac{3}{4}$, then*

(A) $abc = a^2$

(B) $a + b + c = \frac{6}{9}$

(C) $b - c = \frac{a}{6}$

(D) $2(c-a) = 3(b-a)$

(E) None of the above is true

	True	False
A	x	
B		x
C		x
D	x	
E		x

(2) S_1 and S_2 are concentric circles with radii 3 cm and 5 cm**

(A) The area of S_1 is $\frac{3}{5}$ of the area of S_2

(B) The circumference of S_2 is $\frac{3}{5}$ of the circumference of S_1

(C) The circumference of S_1 twice the diameter of S_2

(D) All tangents to S_2 are chords of S_1

(E) Any chord of S_2 which is tangent to S_1 is 8 cm. long

	True	False
A		x
B		x
C	x	
D		x
E	x	

One of the criticisms of the method was that there was a guessing element involved in True-False answers. These criticisms were not entirely met by scoring the results of the test by means of a formula involving Right-Wrong answers. However, the True/False element in the multi-facet situation is not essential - multi-facet questions could just as easily be phrased in such a way as to require the answers to be found and stated in a conventional way. For example***

(1) You are given the following information:

$a = \frac{1}{6}$, $b = 0.60$; $c = \frac{2}{9}$; $d = 0.56$;

$e = \frac{1}{4}$; $f = 0.64$

(A) Place a, b, c, d, e, f, in order of size, the greatest first.

(B) Find the value of $(c-a)-(e-c)$

Answers

*Secondary School Examinations Council: The Certificate of Secondary Education: Experimental Examinations - Mathematics (Examinations Bulletin No. 2), London, H.M.S.O., 1964, p. 20.

**Ibid.

***Bulletin No. 7.

(C) Is the average value of b,d, and f equal to, greater than or less than the average value of d and f?

(D) How great is the difference between $\frac{a}{c}$ and $\frac{d}{f}$?

(E) What is the value of the product abd?

Answers

The Psychometric Movement

The word 'test' has been avoided in describing the objective-type examination because, in addition to its usual meaning of an examination, it has a technical meaning[1] of a set of multi-choice and multi-facet objective-type questions constructed by persons technically trained in mental testing, statistics and psychometric techniques. The items of the test are thoroughly tried out and standardised before the test is administered to candidates. A test may be either psychological or educational. A psychological test assesses such abilities as general intelligence, verbal reasoning, spatial judgment, mechanical aptitude and so on. An educational test assesses achievement in subjects or technical skills. An educational test looks like what any teacher might prepare as an objective-type examination. In fact, a standardised test requires time and technique for construction. Measures of item difficulty and item discrimination enable the test constructor to construct tests which are technically efficient in that they discriminate between candidates and they are reasonably reliable and consistent. Providing the test constructor also considers carefully his aims and objectives, prepares a blue-print which makes sure that his aims are realised through an adequate sampling of the course of instruction, a thoroughly efficient test can be constructed. It is however very important to keep in the forefront of one's minds the aim of the test. If it is required to differentiate between candidates over a large range of ability then each item should discriminate. In a short test we cannot afford the luxury of items which all candidates either answer correctly or incorrectly. Hence the concern of the test constructors for item statistics. But not all tests, and certainly not all examinations, are of this kind. A teacher may need to use a test to see whether what he has taught has been understood and assimilated. In particular, there will be certain key operations which must be performed automatically and accurately. In the stress on teaching for understanding the need for automatic and accurate response is sometimes forgotten. In testing for such qualities we want a completely correct response from all our pupils. We are thus quite content to have items which do not discriminate and do not have any incorrect answers. In this situation, which ought to be common in teaching, item statistics are almost irrelevant. Similarly in curriculum evaluation we may be interested to know whether or not the children have understood key concepts. Here we may need items which measure understanding of these concepts. The items assembled for this purpose may be of the kind in which the response is either correct or incorrect and a whole test could be constructed so that something like a 90% correct response is required if the teaching and learning situation can be considered

[1] Vernon, P.E.: The Certificate of Secondary Education - an introduction to objective-type examinations (Examinations Bulletin No.4), London H.M.S.O., 1964, p.2.

satisfactory. Similar considerations apply in programmed learning. The moral of all this is that the whole apparatus and theory of mental testing must be looked at carefully and the aims of the test or the examination must never be forgotten.

The Test Development and Research Office of the West African Examinations Council, Lagos, Nigeria, constructs tests in mathematics and other subjects for the selection examinations and the West African School Certificate and the G.C.E. examinations.

Curriculum Evaluation

The pattern of curriculum development in both the United States and the United Kingdom over the past decade has been one of acts of faith and trial and error. This has been particularly so in mathematics and science. Men and women of ability and vision have become increasingly dissatisfied with the content of traditional mathematics both as regards its intrinsic mathematical value in this modern age and for its impact in terms of relevance, interest and difficulty for the pupils in the schools. They have had an unshakeable belief that new content should be introduced, and they have gone ahead and experimented with it. Sometimes the most interesting new ideas and new content have been suggested by professional mathematicians with little experience in the classroom. In Great Britain, the unreality which might stem from a lack of practical experience of children has been avoided by organising elaborate and extensive trials of the new materials and the new methods in school. Many teachers and many schools have been brought in to the developments, and arrangements were made by feed-back procedures to ensure that the lessons learned were used to modify the new materials. There has been used a system of trial and error that is in itself a kind of curriculum evaluation which should not be despised or under-valued. There is certainly need to go a good deal further. Curriculum development projects cost money and it is essential to know whether the money has been well spent and how to spend money more efficiently on the next project. It is important that the effects of the new curricula on the lives and experiences of the children are healthy.

In a new development in curriculum, the general and the particular aims and objectives of the development should be set out. The specific objectives which follow from the general aims should be interpreted in terms of behaviour. Once the behavioural changes are known, test situations can be devised to measure the change. Thus the success of the curriculum in terms of the developers' own aims and objectives can be measured. It is then a separate issue, best done independently, to evaluate the intrinsic value of the original intention of the project as a whole. It is becoming usual to attach to any curriculum development team a psychometrician whose role is to ask pertinent questions of his colleagues, to get them to clarify their aims and objectives, to organise tests of pupils' understanding and of changes in behaviour, and to make sure that the feed-back of information derived from trials is efficient, adequate, and acted upon. Curriculum development is becoming a feature of educational development in many countries. It is necessary to evaluate the new curricula and those engaged in these projects would find useful the experience of similar project teams, such as the Nuffield Ordinary Level G.C.E. Science Development and the Nuffield Primary Mathematics Project.

Conclusion

Examinations, tests, assessments, and the evaluation of curricula are all inter-related. In assessing and examining children we are to some extent evaluating our curricula, new and old, and we are also measuring our success as teachers. Examinations, both external and internal, still have an important part to play in the maintenance of standards, in the guidance of teachers, perhaps in acting as incentives for pupils, and for qualifying and selection purposes. The best examinations in the future will be a compromise between the 'teach and test' procedures which all good teachers need to employ, and the external checks which are still necessary. The new examinations must not be allowed to inhibit the most enterprising teaching and they must allow intrinsic good teaching of all that is best in mathematics to take place. The best psychometric procedures should be adopted to make the examination as reliable and valid as possible. In curriculum evaluation, we should strive to make our evaluations as scientific as possible but should recognise also the merit of rough and ready procedures. What we need are inspired examiners and imaginative evaluators. The most hopeful long term solution to the problem is to make the teachers and the testers coincide in the same people, and similarly to combine the developers and the evaluators if not into the same people at least into the same team of people.

Suggestions for Action

1. Whatever the programme, it is important for the teacher to assess the progress of the pupils, test their knowledge and skills from time to time and as far as possible ascertain their potential and attitudes. Therefore the teacher must keep himself up-to-date in the methods of examinations, tests and assessments of pupils and evaluation of programmes.

2. In order that the assessment of pupils may be meaningful, the teacher should understand the curriculum and its objectives and preferably participate in the construction and review. He should therefore demonstrate concern and interest by making the assessment and evaluation and feeding back to the curriculum development team.

3. Teachers' professional associations can influence examination practices and curriculum changes. Therefore a teacher should belong to his professional association.

4. Teachers generally and primary school teachers in particular must try to ensure that the child is ready to go on to the next step or to a new experience.

5. It is important that an understanding of mathematical language be properly assessed, especially with young children.

6. Continual evaluation by the teacher is of the utmost importance and records should be kept. Not only will this help the teacher to assess the pupils but also to assess his own procedures.

7. Where a qualifying or selective test has to be set externally it is recommended that teachers should be involved as much as is possible in devising the content and method of examination.

8. In countries where a selective or an achievement test is necessary, such a test should include multiple-choice, multi-facet, and traditional types of items. Where possible, course work and practical tests should also be included. Where teachers are able, it is desirable that teacher opinion be given high rating in the assessment of students.

9. Overall aims, as well as aims within stages of a programme, should be clearly stated and the programme or project can then be evaluated in terms of those objectives. A sharing of opinions about and comparison of results from similar programmes in different countries is to be encouraged. For this purpose, a teacher or his school should subscribe to one or two mathematical journals from other countries.

10. Teachers should avail themselves of in-service courses in examinations, tests, assessments and evaluation techniques.

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6. TEACHERS: INITIAL AND SUBSEQUENT TRAINING

Introduction

There is a shortage of competent teachers of mathematics at all levels. In some countries, the shortage is of qualified teachers; in all there is a dearth of good teachers, competent in action. In several developing countries, there are many teachers who have not been exposed to the new ideas and yet have to teach mathematics. In particular, there are primary school teachers who have little or no knowledge of the subject matter or of the approach which is now recommended in the teaching of mathematics in the primary school. There are tutors in the training colleges who themselves need to become acquainted with the new ideas and approach.

Throughout the world there is an increasing demand for mathematicians in administration, industry and commerce, where conditions of service are more attractive. The status and salary of the school teacher are less favourable than that accorded to comparable positions open to mathematicians outside teaching. The role of a school teacher appears to have little attraction for the high ability group, some of whose members might make good teachers of mathematics.

In developing countries some teachers receive no initial training at all, because of the expense to the countries not only of the training itself but also of the higher salary scale on which trained teachers must be paid. In others, mere shortage of teachers makes it unrealistic to demand initial training. There is also some 'brain-drain' arising from students who find their way to developed countries, achieve higher qualifications, and remain in those countries where conditions of employment are more favourable. Loss also occurs when qualified teachers of mathematics now in service in their own countries seek more lucrative forms of employment. However, there is a core of able and competent men and women in every country who have remained in the profession despite financial loss. Efforts should be made to examine further the reasons for the shortage of competent teachers of mathematics and practical steps taken to ensure a good supply of such teachers and to retain them in the profession.

Careful consideration should be given to the training of those students who, although they have so far not demonstrated the highest mathematical ability, have the qualities required of prospective teachers. Nevertheless teachers of mathematics must know mathematics and how to teach it; they must be competent to continue learning and doing mathematics.

Entry Qualifications

Entry qualifications into the training colleges for primary school teachers vary. In some countries a secondary school certificate (usually following eleven years of successful schooling) is required; in others slightly lower qualifications are required and the length of professional training is prolonged to three or four years. In other countries, the Primary School Leaving Certificate (obtained after six to eight years of schooling) is the required minimum qualification. A desirable entry qualification at which to aim for all teachers is the Secondary School Certificate or passes in five subjects at the General Certificate of Education (Ordinary Level). A pass in mathematics, while desirable, is not essential.

In countries where the average student applying to enter the Training College does not possess the desired minimum entry qualification, it is all the more important to have tutors competent to handle mathematics in the manner recommended in this Chapter. The experience of some countries shows that candidates who are older and have had some responsible experience since leaving school, are likely to prove good candidates for a course of training. But this should not be accepted as a rule, applicable everywhere.

Professional training for university graduates is highly desirable. Among other things, it assists them to form good attitudes to their profession, to learn about children and their ways of learning and to communicate meaningfully and effectively with their pupils. It should lead to a deeper understanding both of the aims of teaching and of the problems usually met in the classroom.

Initial Training

Training in colleges for primary school teachers usually combines professional training with academic study of a number of subjects including mathematics. There should be a mathematics course for all students which would give them a new insight into the mathematics that they would be expected to teach and the methods they might use, and a familiarity with the experiences through which children learn. There should also be an optional specialist course designed to stimulate students in their own mathematical pursuits.

At least some training colleges should provide courses for students without a university degree who intend to teach mathematics at a secondary school. Such a course could be a straight three-year course (as in some countries), which might lead on, immediately or at a later stage, to a Degree Course; or it could be a two-year course followed by a Mathematics Specialist Course of one year. Other variations are possible. Courses might be organised in which students specialise in two main subjects, one of which might be mathematics and the other a language (English, French, a local language, etc.) or some other Arts or Science subject.

There are two schemes for training university graduates to teach mathematics. The first is a degree followed by professional training; the other is a degree incorporating professional training such as B.Ed., B.A. (Education) or B.Sc. (Education), where the precise qualification depends on the university regulations applicable to the candidate. Each scheme has its advantages and both schemes are already being offered in a number of Universities and Institutes of Education. Some degree courses might be structured to cover mathematics and some Arts subject in order to offset the shortage caused when science graduates with mathematics move into other forms of employment.

Consultation between mathematics tutors in different colleges and countries is highly desirable. It is therefore suggested that opportunities be given to those teaching mathematics at colleges to benefit from bursaries or study leave periodically, in order to study new methods and new topics in teaching mathematics.

Suggestions for Action

1. A teacher, however well qualified in his subject or subjects, should train as a teacher. He thereby enhances his own competence and his status in the profession.

2. A teacher should join the teachers' association and avail himself of the opportunity of professional improvement offered by the association.

3. There is usually in every country a Mathematical Association and/or an association of teachers of mathematics. A teacher should at least join the more relevant association both for the contribution he can make to the association and for his self-improvement.

4. Mathematical journals and bulletins are sources of up-to-date activities and information. A teacher should subscribe to one journal or bulletin at least. Suggestions could be made to Libraries to carry suitable mathematical journals, bulletins and magazines.

5. A teacher should educate himself widely by reading mathematics books other than textbooks. There is an increasing number of cheap popular mathematics books written by eminent authors.

6. Mathematics teachers could inspire their pupils by organising mathematics clubs whose activities include excursions, mathematics evenings, games and so on, in which mathematicians from outside the school participate. That is a convenient way of introducing to the pupils mathematicians in industry, technology, administration and business and of widening the education of the pupils in employment opportunities.

7. A good teacher of mathematics must broaden his own education if he is to bring adaptation and relevance to his curriculum and teaching.

8. The primary school teacher, who handles all the subjects in his class, should be familiar with the environment, his children's interests and should have an eye for mathematics in everyday life.

9. A teacher should train and re-train. Opportunities for re-training are provided by one or more members of a University Department or Institute of Education, Ministry of Education, Local Authority, a professional association.

10. Research is not a preserve of University teachers. Simple observation, investigation and records by primary and secondary school teachers could lead to improvement in the teaching and learning of mathematics.

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7. RESOURCES FOR LEARNING MATHEMATICS[1]

The Environment

Resources for learning mathematics are part of our total environment, natural and man-made. The "mathematical content" of our natural environment is much the same for all of us, wherever we happen to be located. The shape of the sun; the apparently changing shape of the moon; daily and seasonal patterns of shadows; regularity and symmetry in crystals, leaves, flowers, and seeds; experiences with water, such as reflections, waves and ripples; - these are freely available to all. We know that many of these things have had a significant effect on the historical development of mathematics, and it is not unreasonable to believe that we should be able to exploit them as resources for the learning of mathematics.

The man-made environment includes all aspects of our total environment which are, directly or indirectly, the creation of man. This includes not only those specific products of our manufacturing technology but also such other man-made creations as our homes; our cities; our social, political and economic systems (including our use of money); our many devices for the transportation of goods, information and ideas; and the many gadgets which flow from the application of our advancing science and technology. The natural environment has played a well-known and significant role in the historical development of mathematics, but the man-made aspects of our environment (and our need to understand and further develop them) have now assumed a far greater significance in relation to current mathematical developments and current curriculum developments.

While we seem to be moving slowly in the direction of greater uniformity in the technological aspects of our societies, there are still very great differences from one place to another. Whether or not this implies the need for differences in mathematics curriculum, there is little doubt that these environmental differences deserve serious consideration in relation to the use of resources for implementing whatever curriculum is used.

The profound influence of our cultural environment on our personal development is, of course, well known to psychologists and anthropologists but we sometimes tend to overlook it in relation to the learning of such a universal and apparently culture-free subject as mathematics. A belief shared by many is that not enough attention has been paid to cultural differences, both in the design of curricula and in the use of various resources for learning mathematics. It might be hopelessly visionary to contemplate a time when the design of mathematics curricula and the use of learning resources will be fully adapted to the needs and potentialities of each individual, but it is not quite so unrealistic to suggest that more attention (especially in the use of resources) should be paid to some of the more obvious cultural differences. This implies, for example, that books designed for use in one society should be "culturally translated", as well as verbally translated, in order to adapt them for the needs of another. Of course, many other dichotomies are possible in addition to such well known ones as rural-urban, affluent-poor, and culturally "normal"-culturally deprived. One which has received a great deal of attention in recent years is related to the relative state of technological advancement, classifying countries rather roughly as "developed" and "developing".

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[1] Adapted largely from the lead paper by Professor A.L. Blakers.

A study[1] by two teachers from one of the most technologically developed countries (the United States of America) in teaching elementary mathematics to children from the Kpelle tribal group in Liberia suggests most strongly that the teacher should understand the culture in which he is working, and how this affects not only the suitability of what is taught (the curriculum) but also the choice of a teaching strategy (including the use of learning resources) which is likely to prove most effective. Another but independent study[2] of social mathematics in a Yoruba society emphasises the need to understand the cultural background and practices of the children in order to assist them effectively in forming mathematical concepts. The Kpelle and the Yoruba tribes are not isolated cases of environmental factors in learning mathematics. The choice of curriculum and the use of available learning resources should take into account the nature, and the changing nature, of the cultural background of the children, and of the future environment in which they will live. If we can discover how to do this successfully, then it will be possible to exploit the environment itself much more than we do as a resource for learning mathematics.

The Teacher

In a list of resources for learning mathematics, the teacher is topmost. The best-conceived curriculum would founder if due attention were not paid to the crucial role of the teacher and to the improvement of teachers themselves, as well as the improvement of other resources such as textbooks and teachers' guides. It is the teacher who is still the main channel for the communication of mathematical ideas to the child, no matter what curriculum is being studied. This statement is as true for classes using the many fine new textbooks that have recently appeared, as it is for classes using older books or having no textbooks at all. And, as far as one can tell at present, the role of the teacher is likely to be just as significant (even if different in detail) in conjunction with such "automated" teaching methods as programmed learning and computer assisted learning. In many ways, the teacher is the critical classroom resource, who sets a limit to the effective use of every other resource - textbooks, films and film strips, programmed materials, attribute blocks, calculators, overhead projectors, number rods and other analogy devices, mathematical games and puzzles, and the many other "teaching aids" which are appearing in our classrooms.

Books

Books are an essential resource for learning mathematics. Books have been with us for a very long time, and their role as carriers of various aspects of our human culture is well known. The two great storehouses for the accumulation and transmission of mathematical knowledge have been people and books. A little reflection shows the extent to which our personal development - in mathematics as in other areas - has been assisted by our use of books.

In the production of books, one interesting development of the last decade (closely associated with curriculum reform) has been the growing tendency for the use of fairly large groups of writers, rather than the more

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1 Gay, J. and Cole, M.: The New Mathematics and an Old Culture, New York, Holt, Rinehart and Winston, 1967.

2 Taiwo, C.O.: Social Mathematics in a Yoruba Society (Unpublished Thesis accepted by the University of London).

traditional authorship by one, or by a small number of writers. This group writing is seen in the work of such bodies as the School Mathematics Study Group, the School Mathematics Project, the African (Entebbe) Mathematics Project, the Joint Schools Project of the Mathematical Association of Ghana, the School Mathematics Project of East Africa and many others. There is no doubt that such group writing can result in a very critical selection and appraisal of material, although it sometimes runs into problems of style. The potentialities of group writing have not been fully realised in most of the recent projects, due to rather severe restrictions of time and money. But, given an adequate supply of both, it seems safe to conjecture that many of the best textbooks of the future will be produced by the co-operative efforts of mathematicians, teachers, learning theorists, and publishers, assisted by feedbacks from the pupils on whom the texts are evaluated.

There have also been significant changes in recent years in the art and technique of book production. Examples are the attractive use of art work and colour in the books of the Nuffield Mathematics Project,[1] the Houghton Mifflin Modern Mathematics series,[2] the New Oxford Junior Mathematics[3] and Freedom to Learn:[4] these books are a far cry from the generally drab textbooks of a generation ago. Another innovation is in the use of transparent overlays. Examples are the School Mathematics Project and the Houghton Mifflin Modern Mathematics series.

Of considerable importance are the increasing number and the widening range of mathematics books which could assist teachers and pupils in learning mathematics. There are the general books, textbooks, reference books, monographs, books on mathematical games, recreations and puzzles, biographical and historical books, journals and bulletins. A number of them, published by one mathematical association or another, are sold at minimal prices. School Libraries and individual teachers may make good collections at a moderate cost. There are useful bibliographies and reviews in journals and bulletins, which supply helpful information for collectors.

In many developing countries, there are few school mathematics textbooks which are written with the background, interest and needs of the children of the respective countries in mind. Some of the so-called adapted textbooks are little more than the foreign books with changed or modified titles. Some of the textbooks in use in some of these countries are old and out-of-date. In countries where a local language other than English is the medium of instruction in the early years at school, mathematics textbooks in the language are extremely few and usually they do not exist. There is therefore need for publishers and authors to co-operate in supplying suitable textbooks in developing countries. There is a place for books on modern methods of teaching mathematics as well as modern content and for background books which would enrich the mathematical education of pupils and teachers.

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[1] Nuffield Mathematics Project: series published by John Murray Ltd., 50 Albemarle Street, London, W.1.

[2] Duncan, Capps, Dolciani, Quast, Zweng: Modern School Mathematics, Structure and Use K-6, Boston, Mass., Houghton Mifflin Company.

[3] Williams, E.M. and James, E.J.: New Oxford Junior Mathematics, London, Oxford University Press, 1971.

[4] Biggs, E.E. and Maclean, J.R.: Freedom to Learn, Don Mills, Ontario, Addison-Wesley (Canada) Ltd., 1969.

Audio-Visual Aids - Films, Film Strips, Television, Radio

Films, film strips, television and radio are resources which involve capital expenditure quite out of the reach of many schools. When they are available they are useful in supplementing the effort of the teacher, compensating for the shortage of qualified teachers and strengthening students and young teachers in their teaching methods.

A slide projector is considered to be of very limited use in the teaching of mathematics and it is being superseded by the overhead projector. The advantages of the overhead projector are

(i) That not only could straight pictures be put on, but a sequence of transparencies could be used to build up a composite picture as a lesson proceeds.

(ii) The structure of such overhead projectors makes it possible for shadows of solid objects to be cast on the screen. In the teaching of motion geometry this would prove a very valuable visual aid.

Television can be used for the following purposes:

(i) To explore regions of mathematics which lend themselves (or could lend themselves) to visual interpretation, especially of a type that teachers cannot easily cope with for one reason or another. The means to this end are very varied but usually involve film animation, electronic wizardry, models and animated captions.

(ii) To explore fresh areas of mathematics with which most pupils and many teachers might not be familiar. This often means re-interpreting rather difficult books and inviting university and college lecturers to the studio. Occasionally it involves extensive actuality filming (e.g. of computers in operation).

(iii) To explore the mathematics all around by drawing on examples from everyday life and by showing applications in science, technology, sociology, etc. Apart from the techniques already mentioned, photographs are often of value here.

(iv) To use the medium to "pipe" all kinds of mathematics to schools which are short of qualified staff. This can allow more effective use of the time of qualified teachers either by themselves or in some form of team-teaching.

(v) To encourage and advance the work of slow learning pupils. Programmes produced by the B.B.C. for slow readers using the full gamut of visual techniques have been particularly successful. One of the most surprisingly successful devices used for encouraging children's work at all levels of ability has been that of showing films of other children working.

(vi) To inform teachers of developments in the subject matter and teaching of the subject using all the techniques mentioned above.

(vii) To act as a source of secondary material in the form of books, films, etc.

Where the facilities exist, video tape may be used in teacher training institutions to enable the students view their own lessons and criticise themselves. It may also be used to record and supplement transmitted programmes.

The use of radio in the teaching of mathematics is of limited value. Mathematics has its language, notation and symbols and lends itself more to visual than to aural presentation. It may however be used with correspondence courses with a view to reinforcing such courses.

One of the advantages of using television and radio is that the wider audience of parents and other interested parties could be reached as well as the pupils and teachers to whom the programme is really directed. This is a great advantage when considering any form of educational change whether of mathematical content or of approach.

Programmed Learning

Programmed learning has made little impact in the teaching of mathematics in developing countries. In the countries where it is being used, it appears not to have lived up to some of the claims made for it. A great deal of experimental work on programmed learning is in progress and may lead to establishing it as a useful resource in supplementing the work of the teacher, but certainly not in replacing him.

Low Cost Teaching Aids and Demonstration Equipment

In the production of low cost teaching aids, emphasis should be on the use of local materials and the low cost. Guidance as to what to make and how to make and use it may be found in the mathematical journals and bulletins of the developed countries. Numbers 18 and 24 of Mathematics Teaching, Journal of Teachers of Mathematics in the United Kingdom, are devoted to structured and other aids.

An important consideration in buying demonstration equipment and gadgets is the cost of buying and maintaining them. In developing countries, this could be prohibitive. Many a time, maintenance service is not adequate or available. A number of these gadgets are made without sufficient precaution against the weathering and deteriorating effect of the tropics. Efforts are being made by Ministries of Education in some countries to set up production centres where suitable aids and equipment are made from local materials and with emphasis on relevance to the local circumstances.

Priorities of Resources

By way of summary, it is desirable to set the priorities in the value of the resources for learning mathematics at the primary and the secondary school levels. In the primary school, the priorities should be

(i) Qualified teachers
(ii) Materials arising from the equipment
(iii) Books (text, reading, reference)

and in the secondary school

(i) Qualified teachers
(ii) Books (various categories comprising text, background, reference, recreations, games, applications)
(iii) Physical equipment to assist investigation and discovery.

Suggestions for Action

1. The teacher is the most important of all the many and varied resources for learning mathematics. He should therefore be well qualified by his initial and continued education in mathematics and the methods of learning and teaching mathematics. He should know the environment of the child and how to present the environment at the different stages of the child's growth and development.

2. The teacher should be able to select the resources that bring excitement, interest and insight into the learning of mathematics by children. The selection should start with the child, things about his person and dress, in the classroom, the school yard, the home, the farm and the street. The child should become a partner in identifying and collecting suitable resources for counting, matching, measuring, weighing and comparing. Examples are chairs, tables, desks, books, pencils, seeds, shells, pebbles and counters, tins, containers of various shapes and sizes, measuring rods, tapes and rulers, string and weights, water and sand. With a variety of commonplace resources, children learn mathematics, not only at school but in the home, the farm and the playground.

3. The value of the resources for learning mathematics is the use that is made of them. The teacher must therefore be familiar with the resources, their use and their contribution to the learning of mathematics. He should then use the resources appropriate to his lesson and suited to the needs of the children.

4. An important resource is the blackboard (black and smooth-surfaced) and a generous supply of white and coloured chalks. These as well as graph boards, geo-boards and mathematical sets for use with the blackboards should be available in the school and used freely.

5. Books are valuable resources for learning mathematics. The teacher himself should cultivate a taste for reading mathematics and inspire his pupils to do the same. The books should range over textbooks, reference books, general books, books on games, recreations, puzzles, applications to other subjects such as science, technology and commerce. To promote education in reading mathematics, the school library should contain a good number and wide range of mathematics books. The teacher should also have a collection, which should include at least one journal or bulletin taken regularly.

6. Textbooks should have relevance to the pupils' background and experience and should cover a wide range of children's activities. This is the more desirable in primary school textbooks, which are an introduction to the reading of mathematics. In addition to the textbooks used by the pupils, the library should have a number of other good textbooks so that the teacher may be aware of other approaches and the pupils of the activities of children in other lands. An example of a good primary school series is the New Oxford Junior Mathematics produced in colours and containing a wide variety of children's activities.

7. Commercially produced structural materials such as Cuisenaire, Dienes', Stern, Structa, Avon, Colour Factor, Unifix and Venture which are used in the primary school for abstracting concepts of

place value and basic number facts and laws are a rare feature of the primary school in developing countries on account of their cost. In the circumstances, teachers in developing countries must concentrate on improvising and making mathematical aids from local materials, with the object of assisting children to make their abstractions.

8. In the secondary school, the workshop, the farm, the play fields, the laboratories and in fact the whole school premises are resources for learning mathematics. It is for the teacher to draw on these resources for his lessons. Examples are geometrical models (made in the workshop), plans, statistics leading to graphs, probability, sets, etc.

9. Audio-visual aids such as television, radio, films, film strips, slides, video tapes, overhead projectors are resources for learning mathematics and other subjects of the school curriculum. The use of these aids is limited by the availability of the facilities (television and radio) and the cost and maintenance of projectors and films, film strips and slides. Where the facilities exist, the resources should be used to reinforce other resources and the teacher. In developed countries and the urban areas of developing countries, television and radio are used in education; films, film strips and slides and even projectors may be borrowed from libraries. In many schools, primary and secondary, this class of aids is yet a remote resource.

Audio-visual aids centres in the University Departments and Institutes of Education and Colleges of Education (Advanced Teachers Colleges) have these aids and give courses on their use and maintenance to students and teachers. Mathematics Teaching No. 24[1] and Freedom to Learn[2] contain assessments of the aids.

10. It is to the teacher we turn for popularising mathematics and finding ways and means of creating the atmosphere for learning mathematics. The organisation of school mathematics clubs, formulation of mathematical projects, teachers' associations, exchange visits, excursions to industrial works, and a number of other activities and ideas depend for their success on the teacher.

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1 Association of Teachers of Mathematics: Mathematics Teaching No.24

2 Biggs, E.E. and Maclean, J.R.: Freedom to Learn: Ontario, Addison-Wesley (Canada) Ltd., 1969.

Selected Reading List

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